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Quasi-Hermitian supersymmetric extensions of a non-Hermitian oscillator Hamiltonian and of its generalizations

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Abstract

A harmonic oscillator Hamiltonian augmented by a non-Hermitian $\mathcal{PT}$-symmetric part and its su(1,1) generalizations, for which a family of positive-definite metric operators was recently constructed, are re-examined in a supersymmetric context. Some quasi-Hermitian supersymmetric extensions of such Hamiltonians are proposed by enlarging su(1,1) to a su(1, 1/1) $\sim$ osp(2/2, $\mathbb{R}$) superalgebra. This allows the construction of new non-Hermitian Hamiltonians related by similarity to Hermitian ones. Some examples of them are reviewed.

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1. Introduction

After the revival of the long-standing interest in non-Hermitian Hamiltonians, which followed the seminal paper of Bender and Boettcher [1] on a class of simple non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians with a real and positive spectrum, many studies have been devoted to extending the framework of $\mathcal{N} = 2$ supersymmetric quantum mechanics (SUSYQM) [2, 3], the related intertwining operator method [4] or the Darboux algorithm [5] to the non-Hermitian sector. SUSYQM has indeed provided a very useful technique for generating a lot of new exactly solvable (or quasi-exactly solvable) non-Hermitian (not necessarily $\mathcal{PT}$-symmetric) Hamiltonians with real (and/or complex) discrete eigenvalues by complexifying the underlying superpotential (see, e.g., [6–11]). Such an approach has also been extended to some higher-order generalizations of SUSYQM (see, e.g., [12–15]).

Non-Hermitian Hamiltonians have been discussed in the general framework of pseudo-Hermiticity, which amounts to the existence in the relevant Hilbert space of a Hermitian invertible operator ζ such that $H^\dagger = \zeta H \zeta^{-1}$ [16]. In such a context, the concept of pseudo-Hermitian SUSYQM has been introduced [16–18] [1] by replacing the superalgebra of standard SUSYQM by

$$\mathcal{Q}^2 = \mathcal{Q}^{\sharp 2} = 0, \qquad \{\mathcal{Q}, \mathcal{Q}^\sharp\} = 2H_{\mathrm{S}}, \tag{1.1}$$

where all operators remain $\mathbb{Z}_2$-graded as usual, but H_{S} is pseudo-Hermitian with respect to some ($\mathbb{Z}_2$-graded) ζ_S, while the supercharges $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ are pseudo-adjoint of one another with respect to the same, i.e., $\mathcal{Q}^\sharp = \zeta_S^{-1}\mathcal{Q}^\dagger\zeta_S$. A generalization of this type of approach to higher order has also been proposed [21, 22].

A consistent unitary theory of quantum mechanics for non-Hermitian Hamiltonians with a real spectrum can be formulated in terms of a positive-definite ζ (here denoted by ζ_+) [16, 23, 24]. Pseudo-Hermiticity is then termed quasi-Hermiticity: the Hamiltonian is Hermitian with respect to a new inner product of the Hilbert space, defined in terms of the metric operator ζ_+, and has a Hermitian counterpart $h = \rho H \rho^{-1}$ (where $\rho = \sqrt{\zeta_+}$) with respect to the original inner product. Similarly, in the SUSYQM approach, for a positive-definite ζ_{S+}, equation (1.1) defines what we may call quasi-Hermitian SUSYQM, the supersymmetric Hamiltonian H_{S} having a ($\mathbb{Z}_2$-graded) Hermitian counterpart $h_{\mathrm{S}} = \rho_{\mathrm{S}} H_{\mathrm{S}} \rho_{\mathrm{S}}^{-1}$ (with $\rho_{\mathrm{S}} = \sqrt{\zeta_{S+}}$).

Many features of non-Hermitian Hamiltonians with a real spectrum have been studied [25–31] on a harmonic oscillator Hamiltonian augmented by a $\mathcal{PT}$-symmetric part, first proposed by Swanson [25]. It is defined by

$$H = \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \alpha a^2 + \beta a^{\dagger 2}, \tag{1.2}$$

where $a^\dagger = (\omega x - \mathrm{i}p)/\sqrt{2\omega}$ and $a = (a^\dagger)^\dagger$ are standard harmonic oscillator creation and annihilation operators (with $p = -\mathrm{i}\mathrm{d}/\mathrm{d}x$ and $\hbar = m = 1$), while ω, α, β are three real parameters such that $\alpha \neq \beta$ and $\Omega^2 = \omega^2 - 4\alpha\beta > 0$. This Hamiltonian is known to have a real, positive and discrete spectrum given by $E_n = \left(n + \frac{1}{2}\right)\Omega$, $n = 0, 1, 2, \ldots$.

For H defined in (1.2), a family of positive-definite metric operators $\zeta_+(z)$, depending on a continuous variable $z \in [-1, 1]$, has been constructed [31] by a generalized Bogoliubov transformation approach. In addition, for each $\zeta_+(z)$ the corresponding equivalent Hermitian Hamiltonian $h(z)$ has been built and an additional observable $O(z)$, forming an irreducible set with H and therefore fixing the choice of the metric [23], has been determined.

In a recent work [32], we have proposed an alternative derivation of these results, based on the fact that the Hamiltonian (1.2) can be written as a linear combination of su(1,1) generators K_0, K_+, K_-, characterized by the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0, \tag{1.3}$$

and the Hermiticity properties

$$K_0^\dagger = K_0, \qquad K_\pm^\dagger = K_\mp. \tag{1.4}$$

One may indeed write

$$H = 2\omega K_0 + 2\alpha K_- + 2\beta K_+, \tag{1.5}$$

with

$$K_0 = \tfrac{1}{2}\left(a^\dagger a + \tfrac{1}{2}\right), \qquad K_+ = \tfrac{1}{2}a^{\dagger 2}, \qquad K_- = \tfrac{1}{2}a^2. \tag{1.6}$$

Our derivation, being independent of the su(1,1) generator realization (1.6), has extended the results of [31] to all Hamiltonians that can be written in the form (1.5) with $\omega, \alpha, \beta \in \mathbb{R}$,

[1] We prefer to use the full name 'pseudo-Hermitian SUSYQM' instead of its often used abbreviated form 'pseudo-SUSYQM' to avoid confusion with a different previous extension of SUSYQM designated by the latter name [19, 20].

$\alpha \neq \beta$ and $\Omega^2 = \omega^2 - 4\alpha\beta > 0$. So we have established that such Hamiltonians admit positive-definite metric operators $\zeta_+ = \rho^2$ with

$$\rho = \exp\{\epsilon[2K_0 + z(K_+ + K_-)]\}, \tag{1.7}$$

where

$$\epsilon = \frac{1}{2\sqrt{1-z^2}} \operatorname{arctanh} \frac{(\alpha-\beta)\sqrt{1-z^2}}{\alpha+\beta-z\omega}, \qquad -1 \leqslant z \leqslant 1,$$

or, in other words,

$$\rho = \left(\frac{\alpha+\beta-\omega z+(\alpha-\beta)\sqrt{1-z^2}}{\alpha+\beta-\omega z-(\alpha-\beta)\sqrt{1-z^2}}\right)^{[2K_0+z(K_++K_-)]/(4\sqrt{1-z^2})}. \tag{1.8}$$

The corresponding equivalent Hermitian Hamiltonian and additional observable are given by

$$h = \frac{1}{2\omega}[\nu(2K_0+K_++K_-)+\mu\omega^2(2K_0-K_+-K_-)] \tag{1.9}$$

and

$$O = 2K_0 + z(K_+ + K_-),$$

respectively. In (1.9), μ and ν are defined by

$$\mu = [(1+z)\omega]^{-1}\left[\omega-(\alpha+\beta)z-(\alpha+\beta-\omega z)\left(1-\frac{(\alpha-\beta)^2(1-z^2)}{(\alpha+\beta-\omega z)^2}\right)^{1/2}\right],$$

$$\nu = (1-z)^{-1}\omega\left[\omega-(\alpha+\beta)z+(\alpha+\beta-\omega z)\left(1-\frac{(\alpha-\beta)^2(1-z^2)}{(\alpha+\beta-\omega z)^2}\right)^{1/2}\right].$$

The aim of the present paper is to provide a quasi-Hermitian supersymmetric extension of the non-Hermitian oscillator Hamiltonian (1.2) and of its generalizations (1.5) valid for any $\zeta_+(z)$. For such a purpose, we shall have to enlarge the su(1,1) Lie algebra considered in [32] to an $\mathrm{su}(1,1/1) \sim \mathrm{osp}(2/2,\mathbb{R})$ Lie superalgebra. This programme is carried out for the former Hamiltonian in section 2 and generalized to the latter one in section 3. Finally, section 4 contains the conclusion.

2. Quasi-Hermitian supersymmetric extension of the non-Hermitian oscillator

Since the Hamiltonian (1.2) has the same spectrum as a harmonic oscillator of frequency Ω, it is clear that we may look for a quasi-Hermitian supersymmetric extension of the type

$$H_{\mathrm{S}} = H_{\mathrm{B}} + H_{\mathrm{F}}, \qquad H_{\mathrm{B}} = H, \qquad H_{\mathrm{F}} = \Omega\left(b^\dagger b - \tfrac{1}{2}\right), \tag{2.1}$$

where $b^\dagger$ and b are some fermionic creation and annihilation operators with $b^2 = b^{\dagger 2} = 0$, $\{b, b^\dagger\} = 1$ and $[b, a] = [b, a^\dagger] = [b^\dagger, a] = [b^\dagger, a^\dagger] = 0$. This Hamiltonian has indeed the same spectrum $\Omega(n_{\mathrm{B}} + n_{\mathrm{F}})$, $n_{\mathrm{B}} = n = 0, 1, 2, \ldots, n_{\mathrm{F}} = 0, 1$, as the boson–fermion harmonic oscillator [3]. It acts on the ($\mathbb{Z}_2$-graded) boson–fermion Fock space $\mathcal{H}_{\mathrm{S}} = \mathcal{H}_{\mathrm{B}} \otimes \mathcal{H}_{\mathrm{F}}$. The metric operator, corresponding to (1.8) and acting on $\mathcal{H}_{\mathrm{B}}$, can be trivially extended to the whole space $\mathcal{H}_{\mathrm{S}}$ and we shall keep the same symbol to denote the extended operator. It is then obvious that the Hamiltonian (2.1) is quasi-Hermitian with respect to the latter and that the equivalent Hermitian supersymmetric Hamiltonian is given by

$$h_{\mathrm{S}} = h_{\mathrm{B}} + h_{\mathrm{F}}, \qquad h_{\mathrm{B}} = h, \qquad h_{\mathrm{F}} = H_{\mathrm{F}}.$$

Such a Hermitian Hamiltonian, with spectrum $\Omega(n_{\mathrm{B}} + n_{\mathrm{F}})$, can be explicitly written as a boson–fermion oscillator by expressing h_{B} as

$$h_{\mathrm{B}} = \Omega\left(\tilde{a}^\dagger \tilde{a} + \tfrac{1}{2}\right),$$

where

$$\tilde{a}^\dagger = \frac{1}{2\sqrt{\omega}} \left\{ \left[\left(\frac{\nu}{\mu}\right)^{1/4} - \omega \left(\frac{\mu}{\nu}\right)^{1/4} \right] a + \left[\left(\frac{\nu}{\mu}\right)^{1/4} + \omega \left(\frac{\mu}{\nu}\right)^{1/4} \right] a^\dagger \right\} \tag{2.2}$$

and $\tilde{a} = (\tilde{a}^\dagger)^\dagger$ satisfy bosonic commutation relations. As is well known [3], h_{S} and the supercharge operators

$$Q = \sqrt{2\Omega}\tilde{a}^\dagger b, \qquad Q^\dagger = \sqrt{2\Omega}\tilde{a} b^\dagger \tag{2.3}$$

satisfy the standard SUSYQM superalgebra

$$Q^2 = Q^{\dagger 2} = 0, \qquad \{Q, Q^\dagger\} = 2h_{\mathrm{S}}.$$

From the mutually adjoint supercharges (2.3), we can now construct supercharges $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ fulfilling equation (1.1) with the pseudo-Hermitian Hamiltonian (2.1) and corresponding to a given ζ_+ in (1.8). It is indeed easy to check that such operators can be expressed as

$$\mathcal{Q} = \rho^{-1} Q \rho, \qquad \mathcal{Q}^\sharp = \rho^{-1} Q^\dagger \rho. \tag{2.4}$$

Then using equations (2.2), (2.3) and the inverse of the generalized Bogoliubov transformation of [31],

$$\rho^{-1} a \rho = \left(\cosh\theta + \epsilon \frac{\sinh\theta}{\theta}\right) a + z\epsilon \frac{\sinh\theta}{\theta} a^\dagger,$$
$$\rho^{-1} a^\dagger \rho = \left(\cosh\theta - \epsilon \frac{\sinh\theta}{\theta}\right) a^\dagger - z\epsilon \frac{\sinh\theta}{\theta} a,$$

with $\theta = |\epsilon|\sqrt{1 - z^2}$, equation (2.4) acquires the form

$$\mathcal{Q} = \sigma W_+ + \tau W_-, \qquad \mathcal{Q}^\sharp = \varphi V_- + \chi V_+, \tag{2.5}$$

where $V_\pm, W_\pm$ are expressed in terms of the original bosonic and fermionic operators $a^\dagger, a, b^\dagger, b$ as

$$V_+ = \tfrac{1}{\sqrt{2}} a^\dagger b^\dagger, \qquad V_- = \tfrac{1}{\sqrt{2}} a b^\dagger, \qquad W_+ = \tfrac{1}{\sqrt{2}} a^\dagger b, \qquad W_- = \tfrac{1}{\sqrt{2}} a b. \tag{2.6}$$

In (2.5), σ, τ, φ and χ are the z-dependent real parameters, given by

$$\begin{aligned}
\sigma &= \frac{1}{\sqrt{\omega}} \left\{ (\omega\sqrt{\mu} + \sqrt{\nu}) \cosh\theta - [(\omega\sqrt{\mu} + \sqrt{\nu}) + z(\omega\sqrt{\mu} - \sqrt{\nu})]\epsilon \frac{\sinh\theta}{\theta} \right\}, \\
\tau &= \frac{1}{\sqrt{\omega}} \left\{ -(\omega\sqrt{\mu} - \sqrt{\nu}) \cosh\theta - [(\omega\sqrt{\mu} - \sqrt{\nu}) + z(\omega\sqrt{\mu} + \sqrt{\nu})]\epsilon \frac{\sinh\theta}{\theta} \right\}, \\
\varphi &= \frac{1}{\sqrt{\omega}} \left\{ (\omega\sqrt{\mu} + \sqrt{\nu}) \cosh\theta + [(\omega\sqrt{\mu} + \sqrt{\nu}) + z(\omega\sqrt{\mu} - \sqrt{\nu})]\epsilon \frac{\sinh\theta}{\theta} \right\}, \\
\chi &= \frac{1}{\sqrt{\omega}} \left\{ -(\omega\sqrt{\mu} - \sqrt{\nu}) \cosh\theta + [(\omega\sqrt{\mu} - \sqrt{\nu}) + z(\omega\sqrt{\mu} + \sqrt{\nu})]\epsilon \frac{\sinh\theta}{\theta} \right\}.
\end{aligned} \tag{2.7}$$

The operators $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ of equation (2.5) can alternatively be expressed in terms of x and p as

$$\mathcal{Q} = \frac{1}{2\sqrt{\omega}}[(\sigma + \tau)\omega x - \mathrm{i}(\sigma - \tau)p], \qquad \mathcal{Q}^\sharp = \frac{1}{2\sqrt{\omega}}[(\varphi + \chi)\omega x + \mathrm{i}(\varphi - \chi)p]. \tag{2.8}$$

As illustrations, let us consider the three specific metric operators discussed in the literature [26–30] and presented in [31], corresponding to $z = 0$, $z = 1$ and $z = -1$, respectively. For each of them, we give below the values of the parameters (2.7) and the realization of the supercharges (2.8), satisfying equation (1.1) with the pseudo-Hermitian Hamiltonian (2.1):

(i) for $z = 0$, $\epsilon = (1/4)\log(\alpha/\beta)$ and thus

$$\sigma = \left(\frac{\beta}{\alpha}\right)^{1/4}\left(\sqrt{\omega + 2\sqrt{\alpha\beta}} + \sqrt{\omega - 2\sqrt{\alpha\beta}}\right),$$
$$\tau = \left(\frac{\alpha}{\beta}\right)^{1/4}\left(\sqrt{\omega + 2\sqrt{\alpha\beta}} - \sqrt{\omega - 2\sqrt{\alpha\beta}}\right),$$
$$\varphi = \left(\frac{\alpha}{\beta}\right)^{1/4}\left(\sqrt{\omega + 2\sqrt{\alpha\beta}} + \sqrt{\omega - 2\sqrt{\alpha\beta}}\right),$$
$$\chi = \left(\frac{\beta}{\alpha}\right)^{1/4}\left(\sqrt{\omega + 2\sqrt{\alpha\beta}} - \sqrt{\omega - 2\sqrt{\alpha\beta}}\right),$$

yielding

$$\mathcal{Q} = \frac{1}{\sqrt{\omega}}\left[\left(\gamma_+\sqrt{\omega + 2\sqrt{\alpha\beta}} - \gamma_-\sqrt{\omega - 2\sqrt{\alpha\beta}}\right)\omega x + \mathrm{i}\left(\gamma_-\sqrt{\omega + 2\sqrt{\alpha\beta}} - \gamma_+\sqrt{\omega - 2\sqrt{\alpha\beta}}\right)p\right],$$
$$\mathcal{Q}^\sharp = \frac{1}{\sqrt{\omega}}\left[\left(\gamma_+\sqrt{\omega + 2\sqrt{\alpha\beta}} + \gamma_-\sqrt{\omega - 2\sqrt{\alpha\beta}}\right)\omega x + \mathrm{i}\left(\gamma_-\sqrt{\omega + 2\sqrt{\alpha\beta}} + \gamma_+\sqrt{\omega - 2\sqrt{\alpha\beta}}\right)p\right],$$

with

$$\gamma_\pm = \frac{1}{2}\left[\left(\frac{\alpha}{\beta}\right)^{1/4} \pm \left(\frac{\beta}{\alpha}\right)^{1/4}\right];$$

(ii) for $z = 1$, $\epsilon = -(\alpha - \beta)/[2(\omega - \alpha - \beta)]$ and thus

$$\sigma = \frac{1}{\sqrt{\omega - \alpha - \beta}}(\Omega + \omega - 2\beta), \qquad \tau = \frac{1}{\sqrt{\omega - \alpha - \beta}}(\Omega - \omega + 2\alpha),$$
$$\varphi = \frac{1}{\sqrt{\omega - \alpha - \beta}}(\Omega + \omega - 2\alpha), \qquad \chi = \frac{1}{\sqrt{\omega - \alpha - \beta}}(\Omega - \omega + 2\beta),$$

yielding

$$\mathcal{Q} = \frac{1}{\sqrt{\omega}}\left(\frac{\Omega + \alpha - \beta}{\sqrt{\omega - \alpha - \beta}}\omega x - \mathrm{i}\sqrt{\omega - \alpha - \beta}\,p\right),$$
$$\mathcal{Q}^\sharp = \frac{1}{\sqrt{\omega}}\left(\frac{\Omega - \alpha + \beta}{\sqrt{\omega - \alpha - \beta}}\omega x + \mathrm{i}\sqrt{\omega - \alpha - \beta}\,p\right);$$

(iii) for $z = -1$, $\epsilon = (\alpha - \beta)/[2(\omega + \alpha + \beta)]$ and thus

$$\sigma = \frac{1}{\sqrt{\omega + \alpha + \beta}}(\Omega + \omega + 2\beta), \qquad \tau = -\frac{1}{\sqrt{\omega + \alpha + \beta}}(\Omega - \omega - 2\alpha),$$
$$\varphi = \frac{1}{\sqrt{\omega + \alpha + \beta}}(\Omega + \omega + 2\alpha), \qquad \chi = -\frac{1}{\sqrt{\omega + \alpha + \beta}}(\Omega - \omega - 2\beta),$$

yielding

$$\mathcal{Q} = \frac{1}{\sqrt{\omega}}\left(\sqrt{\omega+\alpha+\beta}\omega x - \mathrm{i}\frac{\Omega-\alpha+\beta}{\sqrt{\omega+\alpha+\beta}}p\right),$$

$$\mathcal{Q}^\sharp = \frac{1}{\sqrt{\omega}}\left(\sqrt{\omega+\alpha+\beta}\omega x + \mathrm{i}\frac{\Omega+\alpha-\beta}{\sqrt{\omega+\alpha+\beta}}p\right).$$

3. Generalized models

Since the operators $V_\pm$, $W_\pm$, defined in (2.6), are the odd generators of an su(1, 1/1) $\sim$ osp(2/2, $\mathbb{R}$) Lie superalgebra [33], whose even generators are K_0, K_+, K_-, defined in (1.6), and

$$Y = \tfrac{1}{2}\left(b^\dagger b - \tfrac{1}{2}\right),$$

we see that the su(1,1) Lie algebra characterizing the non-Hermitian oscillator (1.2) has been naturally extended to such a superalgebra when going to the quasi-Hermitian SUSYQM description (1.1). The quasi-Hermitian supersymmetric Hamiltonian (2.1) can indeed be written as a linear combination

$$H_{\mathrm{S}} = 2\omega K_0 + 2\alpha K_- + 2\beta K_+ + 2\Omega Y \tag{3.1}$$

of the superalgebra even generators, while $\mathcal{Q}$ and $\mathcal{Q}^\sharp$, as given in (2.5), are expressed in terms of the odd generators.

For completeness' sake, let us recall that the nonvanishing commutation or anticommutation relations of K_0, $K_\pm$, Y, $V_\pm$ and $W_\pm$ [34] are given by (1.3) and

$$\begin{aligned}
&[K_0, V_\pm] = \pm\tfrac{1}{2}V_\pm, && [K_0, W_\pm] = \pm\tfrac{1}{2}W_\pm,\\
&[K_\pm, V_\mp] = \mp V_\pm, && [K_\pm, W_\mp] = \mp W_\pm,\\
&[Y, V_\pm] = \tfrac{1}{2}V_\pm, && [Y, W_\pm] = -\tfrac{1}{2}W_\pm,\\
&\{V_\pm, W_\pm\} = K_\pm, && \{V_\pm, W_\mp\} = K_0 \mp Y,
\end{aligned} \tag{3.2}$$

while their Hermiticity properties are provided by (1.4) and

$$Y^\dagger = Y, \qquad V_\pm^\dagger = W_\mp. \tag{3.3}$$

We are now in a position to generalize the results of section 2 to all ($\mathbb{Z}_2$-graded) Hamiltonians that can be written in the form (3.1), where the realization of the su(1, 1/1) even generators K_0, K_+, K_- and Y may be arbitrary. From [32], it follows that such Hamiltonians are quasi-Hermitian with respect to the (trivially extended) positive-definite metric operator $\zeta_+ = \rho^2$ with ρ given in (1.8). We plan to show that there exist supercharges $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ satisfying equation (1.1) with them and related to the same ζ_+. Furthermore, $\mathcal{Q}$ and $\mathcal{Q}^\sharp$ can be expressed in the form (2.5), where $V_\pm$ and $W_\pm$ are the superalgebra odd generators in the corresponding realization, while σ, τ, φ, χ are given by equation (2.7). To prove this assertion, we may only use the superalgebra defining relations (1.3), (1.4), (3.2) and (3.3).

To start with, it is obvious that the first part of equation (1.1) is trivially satisfied. On inserting equations (2.5) and (3.1) in the second part of the same and using equation (3.2), we get the following four conditions on the parameters σ, τ, φ, χ:

$$\sigma\varphi + \tau\chi = 4\omega, \qquad \sigma\varphi - \tau\chi = 4\Omega, \qquad \sigma\chi = 4\beta, \qquad \tau\varphi = 4\alpha.$$

It is then straightforward to check that these restrictions are fulfilled by the expressions given in (2.7).

It remains to show that $\mathcal{Q}^\sharp = \zeta_+^{-1}\mathcal{Q}^\dagger\zeta_+$, which due to (3.3) amounts to the condition

$$\rho(\varphi V_- + \chi V_+)\rho^{-1} = \rho^{-1}(\sigma V_- + \tau V_+)\rho \tag{3.4}$$

for real parameters $\sigma, \tau, \varphi, \chi$. In [32], we proved that ρ, defined in (1.7), can be factorized in either of the following two forms:

$$\rho = \exp(pK_+)\exp(qK_0)\exp(pK_-) = \exp(p'K_-)\exp(q'K_0)\exp(p'K_+),$$

where[2]

$$\begin{aligned} &\mathrm{e}^{-q/2} = \cosh\theta - \epsilon\frac{\sinh\theta}{\theta}, \qquad && p = \frac{z\epsilon\sinh\theta/\theta}{\cosh\theta - \epsilon\sinh\theta/\theta}, \\ &\mathrm{e}^{q'/2} = \cosh\theta + \epsilon\frac{\sinh\theta}{\theta}, \qquad && p' = \frac{z\epsilon\sinh\theta/\theta}{\cosh\theta + \epsilon\sinh\theta/\theta}. \end{aligned} \tag{3.5}$$

On combining these expressions with the Baker–Campbell–Hausdorff formula and equation (3.2), it is then straightforward to prove the relations

$$\begin{aligned} &\rho V_+ \rho^{-1} = \mathrm{e}^{q'/2}(V_+ + p'V_-), \qquad && \rho V_- \rho^{-1} = \mathrm{e}^{-q/2}(V_- - pV_+), \\ &\rho^{-1} V_+ \rho = \mathrm{e}^{-q/2}(V_+ - pV_-), \qquad && \rho^{-1} V_- \rho = \mathrm{e}^{q'/2}(V_- + p'V_+). \end{aligned} \tag{3.6}$$

Finally, equations (3.5) and (3.6) allow us to easily check that condition (3.4) is fulfilled by the parameters given in (2.7). This completes the extension of the results of section 2 to the generalized Hamiltonians (3.1) with equivalent Hermitian Hamiltonians expressed as $h_{\mathrm{S}} = h + 2\Omega Y$, where h is given in (1.9).

Let us now review a few examples of generalized models based on (3.1). This amounts to considering some physically relevant realizations of the su(1, 1/1) generators.

For completeness' sake, let us mention some trivial extensions of the realization considered in section 2. The first one consists in replacing the fermionic creation and annihilation operators $b^\dagger, b$ by 2×2 matrices $\sigma_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\sigma_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. In the second one, instead of a single pair of bosonic and fermionic operators $a^\dagger, a, b^\dagger, b$, one uses n commuting pairs $a_i^\dagger, a_i, b_i^\dagger, b_i, i = 1, 2, \ldots, n$, thus yielding [33]

$$\begin{aligned} &K_0 = \frac{1}{2}\left(\sum_i a_i^\dagger a_i + \frac{n}{2}\right), \qquad && K_+ = \frac{1}{2}\sum_i a_i^{\dagger 2}, \qquad && K_- = \frac{1}{2}\sum_i a_i^2, \\ &Y = \frac{1}{2}\left(\sum_i b_i^\dagger b_i - \frac{n}{2}\right), \qquad && V_+ = \frac{1}{\sqrt{2}}\sum_i a_i^\dagger b_i^\dagger, \qquad && V_- = \frac{1}{\sqrt{2}}\sum_i a_i b_i^\dagger, \\ &W_+ = \frac{1}{\sqrt{2}}\sum_i a_i^\dagger b_i, \qquad && W_- = \frac{1}{\sqrt{2}}\sum_i a_i b_i. \end{aligned} \tag{3.7}$$

The resulting pseudo-Hermitian supersymmetric Hamiltonian (3.1) is then a non-Hermitian n-dimensional boson–fermion oscillator.

A somewhat more subtle realization uses spin operators and provides a non-Hermitian generalization of the three-dimensional harmonic oscillator with a constant spin–orbit coupling $\frac{1}{2}(\boldsymbol{p}^2 + \omega^2\boldsymbol{x}^2) \pm \omega\left(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \frac{3}{2}\right)$, first studied by Ui and Takeda [35] and whose 'accidental' degeneracies were explained by Balantekin [36]. In this case, the realization of the su(1, 1/1) generators reads

$$Y = \frac{1}{2}\left(\sum_i \sigma_i L_i + \frac{3}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sum_i \sigma_i a_i^\dagger \\ 0 & 0 \end{pmatrix}, \quad V_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sum_i \sigma_i a_i \\ 0 & 0 \end{pmatrix},$$

$$W_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \sum_i \sigma_i a_i^\dagger & 0 \end{pmatrix}, \quad W_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \sum_i \sigma_i a_i & 0 \end{pmatrix}$$

with K_0, K_+, K_- as in (3.7) and i running over 1, 2, 3.

[2] In these equations, we correct a misprint in the corresponding result of [32] ($\mathrm{e}^{-q'/2}$ is replaced by $\mathrm{e}^{q'/2}$).

Turning ourselves now to the rich realm of generalized Calogero models [37], which were already signalled in [32] as a possible domain for useful applications of our su(1,1) approach to [31], we may consider $\mathcal{N} = 2$ superconformal extensions of such models [38, 39]. In the case considered by Freedman and Mende [38], for instance, the su(1, 1/1) generators are given by

$$K_0 = \frac{1}{2\omega}\left(-\frac{1}{2}\sum_i \nabla_i^2 + \frac{1}{2}g^2 \sum_{\substack{i,j\\ i\neq j}} \frac{1}{(x_i - x_j)^2} + \frac{1}{2}\omega^2 \sum_i x_i^2 + \frac{1}{2}g \sum_{\substack{i,j\\ i\neq j}} \frac{1}{(x_i - x_j)^2}[b_i^\dagger, b_j - b_i]\right),$$

$$K_\pm = \frac{1}{2\omega}\left[\frac{1}{2}\sum_i \nabla_i^2 - \frac{1}{2}g^2 \sum_{\substack{i,j\\ i\neq j}} \frac{1}{(x_i - x_j)^2} + \frac{1}{2}\omega^2 \sum_i x_i^2 \mp \omega\left(\sum_i x_i\nabla_i + \frac{n}{2}\right)\right.$$
$$\left. - \frac{1}{2}g \sum_{\substack{i,j\\ i\neq j}} \frac{1}{(x_i - x_j)^2}[b_i^\dagger, b_j - b_i]\right],$$

$$Y = \frac{1}{4}\left(\sum_i [b_i^\dagger, b_i] + gn(n-1)\right),$$

$$V_\pm = \frac{1}{2\sqrt{\omega}}\sum_i\left(\mp\nabla_i + \omega x_i \mp g\sum_{j\neq i}\frac{1}{x_i - x_j}\right)b_i^\dagger,$$

$$W_\pm = \frac{1}{2\sqrt{\omega}}\sum_i\left(\mp\nabla_i + \omega x_i \pm g\sum_{j\neq i}\frac{1}{x_i - x_j}\right)b_i,$$

where $x_i, i = 1, 2, \ldots, n$, denote the coordinates of n particles on a line and $b_i^\dagger, b_i, i = 1, 2, \ldots, n$, are n independent pairs of fermionic creation and annihilation operators. The corresponding Hermitian supersymmetric Hamiltonian reads $2\omega(K_0 + Y)$. The analysis presented in this section extends this operator to non-Hermitian Hamiltonians (3.1) with Hermitian counterparts.

4. Final remarks

In the present paper, by using an $su(1, 1/1) \sim osp(2/2, \mathbb{R})$ approach, we have further extended the set of non-Hermitian Hamiltonians to which the family of positive-definite metric operators constructed in [31] for a non-Hermitian oscillator can be applied. This has enlarged the class of non-Hermitian Hamiltonians related to Hermitian ones by similarity transformations and therefore endowed with solid physical foundations.

In particular, we have proposed non-Hermitian generalizations of the Ui and Takeda model and of the Freedman and Mende $\mathcal{N} = 2$ superconformal extension of the Calogero model. These are but some simple examples of relevance of our new proposal, which will undoubtedly prove rich in applications to many fields.

As a final point, it is worth commenting on some specific physical aspects of the non-Hermitian Hamiltonians considered here, which may shed some light on the algebraic construction just carried out. In the literature, one can find other non-Hermitian Hamiltonians with a harmonic oscillator spectrum, for which the non-Hermiticity lies in a complex-valued potential which is purely local. Such is the case, for instance, for some $\mathcal{PT}$-symmetric (or non-$\mathcal{PT}$-symmetric) Hamiltonians constructed by using a SUSYQM or Darboux technique [6]. In contrast, for the non-Hermitian oscillator Hamiltonian (1.2), the non-Hermiticity is due to a momentum-dependent linear interaction $\frac{1}{2}\mathrm{i}(\alpha - \beta)(xp + px)$. The gauge-type transformation performed by ρ, defined in (1.7),[3] eliminates such a non-locality while restoring the Hermiticity. So we may say that for such a Hamiltonian cure for non-Hermiticity is equivalent to cure for non-locality. Of course, the underlying su(1,1) symmetry is largely instrumental in this matter. Such remarks remain true for the supersymmetric extension (2.1) (as well as for the generalizations (3.1) of the latter) because it only differs from (1.2) by a Hermitian term $2\Omega Y$.

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[3] It should be noted that ρ itself depends on p for $z \neq 1$, since equation (1.7) can be rewritten as $\rho = \exp\{\epsilon[(1-z)p^2 + (1+z)\omega^2 x^2]/(2\omega)\}$.

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